# On the Singularity of Multivariate Hermite Interpolation *

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**Abstract**

In this paper we study the singularity of multivariate Hermite interpolation of type total degree. We present a method to judge the singularity of the interpolation scheme considered and by the method to be developed, we show that all Hermite interpolation of type total degree on $m = d + k$ points in $\mathbb{R}^d$ is singular if $d \geq 2k$. And then we solve the Hermite interpolation problem on $m \leq d + 3$ nodes completely. Precisely, all Hermite interpolations of type total degree on $m \leq d + 1$ points with $d \geq 2$ are singular; for $m = d + 2$ and $m = d + 3$, only three cases and one case can produce regular Hermite interpolation schemes, respectively. Besides, we also present a method to compute the interpolation space for Hermite interpolation of type total degree.

**Keywords:** Hermite interpolation; Singularity; Interpolation space; Polynomial ideal

# 1 Introduction

Let $\Pi^d$ be the space of all polynomials in $d$ variables, and let $\Pi^d_n$ be the subspace of polynomials of total degree at most $n$. Let $\mathscr{X} = \{X_1, X_2, \ldots, X_m\}$ be a set of pairwise distinct points in $\mathbb{R}^d$ and $\mathbf{p} = \{p_1, p_2, \ldots, p_m\}$ be a set of $m$ nonnegative integers. The Hermite interpolation problem to be considered in this paper is described as follows: Find a (unique) polynomial $f \in \Pi^d_n$ satisfying

$$\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_d}}{\partial x_1^{\alpha_1}\ldots\partial x_d^{\alpha_d}} f(X_q) = c_{q,\boldsymbol{\alpha}}, \quad 1 \leq q \leq m, \quad 0 \leq |\boldsymbol{\alpha}| \leq p_q, \tag{1}$$

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for given values $c_{q,\boldsymbol{\alpha}}$, where the numbers $p_q$ and $n$ are assumed to satisfy

$$\binom{n+d}{d} = \sum_{q=1}^{m} \binom{p_q+d}{d}. \tag{2}$$

Following [11, 12], such kind of problem is called Hermite interpolation of type total degree. The interpolation problem $(\mathbf{p}, \mathscr{X})$ is called regular if the above equation has a unique solution for each choice of values $\{c_{q,\boldsymbol{\alpha}}, 1 \leq q \leq m, 0 \leq |\boldsymbol{\alpha}| \leq p_q\}$. Otherwise, the interpolation problem is singular. As shown in [7], the regularity of Hermite interpolation problem $(\mathbf{p}, \mathscr{X})$ implies that it is regular for almost $\mathscr{X} \subset \mathbb{R}^d$ with $|\mathscr{X}| = m$.

**Definition 1** ([7]). *We say that the interpolation scheme* $\mathbf{p}$ *is:*

- *Regular if the problem* $(\mathbf{p}, \mathscr{X})$ *is regular for all* $\mathscr{X}$*.*
- *Almost regular if the problem* $(\mathbf{p}, \mathscr{X})$ *is regular for almost all* $\mathscr{X}$*.*
- *Singular if* $(\mathbf{p}, \mathscr{X})$ *is singular for all* $\mathscr{X}$*.*

The special case in which the $p_q$ are all the same is called uniform Hermite interpolation of type total degree. In the case of uniform Hermite interpolation of type total degree, Eq. (2) should be changed to

$$\binom{n+d}{d} = m\binom{p+d}{d}. \tag{3}$$

The research of regularity of multivariate Hermite interpolation is more difficult that Lagrange case, although the latter is also difficult. One of the main reasons is that Eq. (2) or (3) do not hold in some cases. Up to now, we have known that all the Hermite interpolation on $m \leq d+1$ points are singular except for Lagrange interpolation, see [9, 11, 12]. Besides, no any other results appeared for $m \geq d+2$. Actually, Hermite interpolation of type total degree on $d+2$ nodes in $\mathbb{R}^d$ are not necessary singular. For more research of this area, we can refer to [1, 3–6, 8–13] and the reference therein.

The main purpose of this paper is to investigate the singularity of Hermite interpolation for $m = d+k$ with $k = 1, 2, 3$. Our method is to construct a polynomial which is a solution of the homogenous interpolation problem. By the presented method, we show that all Hermite interpolation of type total degree on $m \leq d+1$ nodes in $\mathbb{R}^d$ are singular except for Lagrange interpolation; on $m = d+2$ nodes in $\mathbb{R}^d$ are singular except for three cases; on $m = d+3$ nodes are singular except for one case. The result of $m \leq d+1$ is well known, but our method is different. Moreover, we also show all the hermite interpolation problem of type total degree with $m = d+k$ nodes are singular for $d \geq 2k$.

This paper is organized as follows. In section 2, we will consider the interpolation space satisfying the Hermite interpolation requirement from the view of polynomial ideal. In this section, Eq. (2) is not required and the polynomial space is not necessary $\Pi_n^d$. In section 3, we consider the singularity of the Hermite interpolation of type total degree and present the main results. Finally, in section 4, we conclude our results.

## 2 Interpolation Space

In this section, we consider the following interpolation problem:

Let $\mathscr{X} = \{X_1, X_2, \ldots, X_m\}$ be a set of pairwise distinct points in $\mathbb{R}^d$ and $\mathbf{p} = \{p_1, p_2, \ldots, p_m\}$ be a set of $m$ nonnegative integers. Find a subspace $\mathcal{P} \subset \Pi^d$ such that for any given real numbers $c_{q,\boldsymbol{\alpha}}, 1 \leq q \leq m, 1 \leq |\boldsymbol{\alpha}| \leq p_q$ there exists a unique polynomial $f \in \mathcal{P}$ satisfying the interpolation conditions

$$\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_d}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} f(X_q) = c_{q,\boldsymbol{\alpha}}, \quad 1 \leq q \leq m, \quad 0 \leq |\boldsymbol{\alpha}| \leq p_q \tag{4}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$ and $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \ldots + \alpha_d$.

Following [14], we call such a pair $\{\mathscr{X}, \mathbf{p}, \mathcal{P}\}$ correct. Clearly, such kind of space $\mathcal{P}$ always exists if no any constraint is added. For the Lagrange case, such kind of interpolation problem was studied extensively by many authors, for example [3, 4, 6, 8] and the reference therein. In [14], Xu presented a solution from the view of polynomial ideal. Now we generalize Xu's result to Hermite case.

Before proceeding, we first present some necessary notations. Throughout of this paper, we use the usual multi-index notation. To order the monomials in $\Pi^d = \mathbb{R}[x_1, \ldots, x_d]$, we use graded lexicographic order. Let $I$ be a polynomial ideal in $\Pi^d$. The codimension of $I$ is denoted by $\mathrm{codim} I$, that is,

$$\mathrm{codim} I = \dim \Pi^d / I.$$

If there are polynomials $f_1, f_2, \ldots, f_r$ such that every $f \in I$ can be written as

$$f = a_1 f_1 + a_2 f_2 + \ldots + a_r f_r, \quad a_j \in \Pi^d,$$

we say that $I$ is generated by the basis $f_1, f_2, \ldots, f_r$, and we write $I = \langle f_1, f_2, \ldots, f_r \rangle$.

For a fixed monomial order, we denote by $LT(f)$ the leading monomial term for any polynomial $f \in \Pi^d$; that is, if $f = c_\alpha X^\alpha$, then $LT(f) = c^\beta X^\beta$, where $X^\beta$ is the leading monomial among all monomials appearing in $f$. For an ideal $I$ in $\Pi^d$ other than $\{0\}$, we denote by $LT(I)$ the leading terms of $I$, that is,

$$LT(I) = \{cX^\alpha | \text{there exists } f \in I \text{ with } LT(f) = cX^\alpha\}.$$

We further denote by $\langle LT(I) \rangle$ the ideal generated by the leading terms of $LT(f)$ for all $f \in I \setminus \{0\}$.

The following theorem is important for out purpose.

**Theorem 1** ([14])**.** *Fix a monomial ordering on $\Pi^d$ and let $I \in \Pi^d$ be an ideal. Then there is an isometry between $\Pi^d / I$ and the space*

$$\mathcal{S}_I := \mathrm{Span}\{X^\alpha | X^\alpha \notin \langle LT(I) \rangle\}.$$

*More precisely, every $f \in \Pi^d$ is congruent modulo $I$ to a unique polynomial $r \in \mathcal{S}_I$.*

Let $I = \langle f_1, f_2, \ldots, f_r \rangle$ and $J = \langle g_1, g_2, \ldots, g_s \rangle$ be two polynomial ideals. The sum of $I$ and $J$, denoted by $I + J$, is the set of $f + g$ where $f \in I$ and $g \in J$. The product of $I$ and $J$, denoted by $I \cdot J$, is defined to be the ideal generated by all polynomials $f \cdot g$ where $f \in I$ and $g \in J$. It is easy to know that $I \cdot J = \langle f_i g_j : 1 \le i \le r, 1 \le j \le s \rangle$. The intersection $I \cap J$ of two ideals $I$ and $J$ in $\Pi^d$ is the set of polynomials which belong to both $I$ and $J$. We always have $I \cdot J \subset I \cap J$. However, $IJ$ can be strictly contained in $I \cap J$. It follows from [2] that if $I$ and $J$ is comaximal, then $IJ = I \cap J$. $I$ and $J$ is comaximal if and only if $I + J = \Pi^d$.

In application, people usually are interested in the space with minimal degree for fixed monomial ordering. For this purpose, consider the following polynomial ideal:

$$I(\mathscr{X}, \mathbf{p}) = \left\{ f \in \Pi^d : \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_d}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} f(X_q) = 0, \quad 1 \le q \le m,\ 0 \le |\boldsymbol{\alpha}| \le p_q \right\} \tag{5}$$

If only one point $X \in \mathscr{X}$ then we can write it as $I(X, p)$.

**Theorem 2.** *Let $I(\mathscr{X}, \mathbf{p})$ be the polynomial ideal defined as above. Then the interpolation problem satisfying Eq.* (4) *has a unique solution in $\mathcal{S}_I$.*

*Proof.* Let $N = \sum_{i=1}^{m} \binom{p_i+d}{d}$. Denote the $N$ linear functionals in (4) by $F_1, F_2, \ldots, F_N$. Thus Eq. (4) can be rewritten as $F_q(f) = c_q$ for $1 \le q \le N$. Suppose $n$ is an integer which is big enough such that there exists a polynomial $f \in \Pi_n^d$ satisfying $F_q(f) = c_q$. We also assume that $\mathcal{S}_I \subset \Pi_n^d$ although it is not necessary for our proof. Thus we have $\Pi_n^d = \mathcal{S}_I \cup (\Pi_n^d \cap I)$. Suppose $\varphi_1, \ldots, \varphi_t$ and $\phi_1, \ldots, \phi_s$ are the basis functions of $\mathcal{S}_I$ and $\Pi_n^d \cap I$, respectively. We further define a column vector

$$\Phi = (\varphi_1, \ldots, \varphi_t, \phi_1, \ldots, \phi_s)^T := (\Phi_1^T, \Phi_2^T)^T.$$

Since $n$ is big enough, we have

$$\mathrm{rank}(F_1(\Phi), F_2(\Phi), \ldots, F_N(\Phi)) = N.$$

Furtherly,

$$\mathrm{rank}(F_1(\Phi_1), F_2(\Phi_1), \ldots, F_N(\Phi_1)) = N$$

since $F_i(\Phi_2) = 0$ for $1 \le i \le N$, which leads to $t \ge N$. It only remains to prove $t \le N$. If $t > N$, then $\mathcal{F}_i := (F_1(\varphi_i), F_2(\varphi_i), \ldots, F_N(\varphi_i)), 1 \le i \le t$ are linearly dependent and there exist scalars $a_1, a_2, \ldots, a_t$, not all zero, such that $\sum_{i=1}^{t} a_i \mathcal{F}_i = 0$. In terms of the components of the vector $\mathcal{F}_i$, this shows that

$$\sum_{i=1}^{t} a_i F_q(\varphi_i) = 0, \quad q = 1, 2, \ldots, N$$

or,

$$F_q\Big(\sum_{i=1}^{t} a_i \varphi_i\Big) = 0, \quad q = 1, 2, \ldots, N.$$

The latter equations means that $\varphi = \sum_{i=1}^{t} a_i \varphi_i \in I(\mathscr{X}, \mathbf{p})$, which is a contradiction to $\varphi \in \mathcal{S}_I$ because every $\varphi_i \in \mathcal{S}_I$. Hence we have $t \le N$ and finally $t = N$, which completes the proof. $\square$

The theorem states that $(\mathscr{X}, \mathbf{p}, \mathcal{S}_I)$ is correct. This result maybe was known more or less, but we did not find it in the literature.

Next, we consider the computation of $I(\mathscr{X}, \mathbf{p})$. If only one point is in $\mathscr{X}$, the result is immediate.

**Lemma 1.** *Let $X$ be a point in $\mathbb{R}^d$ and $p$ be a nonnegative integer, then*

$$I(X,p) = \Big\langle l_1 l_2 \dots l_{p+1} : l_i \text{ is a linear polynomial vanishing at point } X \Big\rangle. \tag{6}$$

For multi-point case, we also have the similar result.

**Theorem 3.**

$$\begin{aligned} I(\mathscr{X}, \mathbf{p}) = \Big\langle f \in \Pi^d : & f \text{ can be divided by the product of } p_i + 1 \\ & \text{linear polynomials which pass through } X_i, i = 1, 2, \dots, m \Big\rangle \end{aligned} \tag{7}$$

*Proof.* Without loss of generality, we only give a proof for $m = 2$, that is, $\mathscr{X} = \{X_1, X_2\}$ and $\mathbf{p} = \{p_1, p_2\}$. Let

$$I(X_1, p_1) = \Big\{ f \in \Pi^d : \frac{\partial^{\alpha_1 + \alpha_2 + \ldots + \alpha_d}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} f(X_1) = 0, \quad 0 \le |\boldsymbol{\alpha}| \le p_1 \Big\},$$
$$I(X_2, p_2) = \Big\{ f \in \Pi^d : \frac{\partial^{\alpha_1 + \alpha_2 + \ldots + \alpha_d}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} f(X_2) = 0, \quad 0 \le |\boldsymbol{\alpha}| \le p_2 \Big\},$$

Obviously $I(\mathscr{X}, \mathbf{p}) = I(X_1, p_1) \cap I(X_2, p_2)$. If we denote by $J$ the right hand of Eq. (7), then we need to show $I(X_1, p_1) \cap I(X_2, p_2) = J$. Obviously $J \subset I(X_1, p_1) \cap I(X_2, p_2)$. Thus it remains to show $J \supset I(X_1, p_1) \cap I(X_2, p_2)$. Notice that $I(X_1, p_1) I(X_2, p_2) \subset J$ holds. Hence the proof will be completed if we can show

$$I(X_1, p_1) \cap I(X_2, p_2) = I(X_1, p_1) I(X_2, p_2).$$

To this end, we will prove that $I(X_1, p_1)$ and $I(X_2, p_2)$ are comaximal, that is, $I(X_1, p_1) + I(X_2, p_2) = \Pi^d$. It is enough to show $1 \in I(X_1, p_1) + I(X_2, p_2)$.

Let $X_1 = (x_1^1, x_2^1, \dots, x_d^1)$ and $X_2 = (x_1^2, x_2^2, \dots, x_d^2)$. Assume $x_1^1 \neq x_1^2$. It is easy to check that $(x_1 - x_1^1)^{p_1+1} \in I(X_1, p_1)$ and $(x_1 - x_1^2)^{p_2+1} \in I(X, p_2)$. If $(x_1 - x_1^1)^{p_1+1}$ and $(x_1 - x_1^2)^{p_2+1}$ are seen as two polynomials with respect to $x_1$, that is, they are taken as univariate polynomials, then the greatest common divisor

$$\gcd((x_1 - x_1^1)^{p_1+1}, (x_1 - x_1^2)^{p_2+1}) = 1,$$

which means that there exist two polynomial $q_1(x_1)$ and $q_2(x_1)$ such that

$$(x_1 - x_1^1)^{p_1+1} q_1(x_1) + (x_1 - x_1^2)^{p_2+1} q_2(x_1) = 1.$$

This complete the proof. $\square$

The following two examples will be mentioned again in next section to show the regularity of Hermite interpolation problem.

**Example 1.** Consider the case of $d = 3$, $m = 5$ and $\mathbf{p} = \{1, 1, 1, 1, 1\}$. Take

$$X_1 = (0, 0, 0),\ X_2 = (1, 0, 0),\ X_3 = (0, 1, 0),\ X_4 = (0, 0, 1),\ X_5 = (x_0, y_0, z_0)$$

and $\mathscr{X} = \{X_1, X_2, X_3, X_4, X_5\}$. With the help of Maple, the Groebner basis of $I(\mathscr{X}, \mathbf{p})$ can be written as

$$\begin{aligned}
\{&f(x, y, z, x_0, y_0, z_0), f(z, y, x, z_0, y_0, x_0), f(x, z, y, x_0, z_0, y_0),\\
&g(x, y, z, x_0, y_0, z_0), g(z, y, x, z_0, y_0, x_0), g(x, z, y, x_0, z_0, y_0),\\
&g(y, x, z, y_0, x_0, z_0), g(z, x, y, z_0, x_0, y_0), g(y, z, x, y_0, z_0, x_0),\\
&h(x, y, z, x_0, y_0, z_0), h(y, x, z, y_0, x_0, z_0), h(y, z, x, y_0, z_0, x_0),\\
&w(x, y, z, x_0, y_0, z_0), w(y, z, x, y_0, z_0, x_0), w(x, z, y, x_0, z_0, y_0)\},
\end{aligned}$$

where

$$\begin{aligned}
f(x, y, z, x_0, y_0, z_0) = &-x_0z_0(z_0 - 1)(x_0z_0 + l(X_5) - y_0)(zy^2 + z^2y - zy)\\
&+2(z_0 - 1)z_0(y_0z_0^2 + x_0z_0^2 - y_0z_0 + x_0y_0z_0 - x_0z_0 + x_0y_0)xyz\\
&-y_0(z_0 - 1)z_0(y_0z_0 + l(X_5) - x_0)(xz^2 + x^2z - xz)\\
&+x_0y_0l(X_5)(z^4 - 2z^3 + z^2) - z_0^2(z_0 - 1)^3(x^2y + xy^2 - xy),\\
g(x, y, z, x_0, y_0, z_0) = &\ x_0z_0(x_0z_0 + z_0 - 1)(zy - zy^2) + z_0^2(z_0 - 1)^2(xy - xy^2 - x^2y)\\
&+z_0(-z_0^2 + 2y_0z_0^2 + 2x_0z_0^2 + 2x_0y_0z_0 + 2z_0 - 3y_0z_0 - 4x_0z_0 - 1 + y_0 + 2x_0)xyz\\
&+x_0l(X_5)z^3y - x_0(x_0z_0^2 + x_0 + y_0 + z_0^2 - 1)z^2y + (xz - x^2z - xz^2)y_0^2z_0^2\\
h(x, y, z, x_0, y_0, z_0) = &\ 2y_0z_0(y_0z_0 - y_0 + x_0y_0 - z_0 + x_0z_0 + 1 - 2x_0)xyz\\
&-y_0z_0^2(z_0 - 1)(xy^2 + x^2y - xy) - y_0^2z_0(y_0 - 1)(xz^2 + x^2z - xz)\\
&-x_0y_0z_0(1 + x_0)(zy^2 + z^2y - zy) + x_0l(X_5)z^2y^2\\
w(x, y, z, x_0, y_0, z_0) = &\ l(X_5)xy^2z + (y_0^2 - y_0^3)(xz^2 + x^2z - xz) - x_0^2y_0(zy^2 + z^2y - yz)\\
&-y_0z_0^2(xy^2 + x^2y - xy) + y_0(2x_0y_0 + 2y_0z_0 + 2x_0z_0 - 3x_0 - 2y_0 - 3z_0 + 2)xyz
\end{aligned}$$

and $l(X_5) = x_0 + y_0 + z_0 - 1$. It is easy to see that if any four nodes do not lie on hyperplane then $S_I = \Pi_3^3$. Hermite interpolation of type total degree is affinely invariant in the sense that if the interpolation is singular or regular. Hence if the given five nodes are in general position, that is, no four nodes lie on a hyperplane, then any four of them can be transformed into $X_1, X_2, X_3, X_4$. This example implies that uniform Hermite interpolation of type total degree on 5 nodes up to order 1 in $\mathbb{R}^3$ is almost regular.

**Example 2.** Consider the case of $d = 3, m = 6$ and $\mathbf{p} = \{3, 3, 3, 3, 3, 3\}$. Take

$$X_1 = (0, 0, 0),\ X_2 = (1, 0, 0),\ X_3 = (0, 1, 0),\ X_4 = (0, 0, 1),\ X_5 = (1, 1, 1),\ X_6 = (2, 1, 1)$$

and $\mathscr{X} = \{X_1, X_2, X_3, X_4, X_5, X_6\}$. With the help of Maple, we have $S_I = \Pi_7^3$. Thus uniform Hermite interpolation of type total degree on 6 points up to order 3 in $\mathbb{R}^3$ is almost regular.

# 3 Singular Interpolation Scheme

In this section, we will investigate Hermite interpolation of type total degree which is singular. Our results covers those appeared in [11], but the method employed here is simpler.

In this section, Eq. (2) is always assumed to hold. In this case, the interpolation space and the set of functionals to be interpolated are affinely invariant.

The following theorem and corollary will give an evaluation of $n$ in Eq. (2).

**Theorem 4.** *Given $\mathscr{X} = \{X_1, X_2, \ldots, X_m\}$ and $\mathbf{p} = \{p_1, p_2, \ldots, p_m\}$, if there exists an $n$ such that $(\mathscr{X}, \mathbf{p}, \Pi_n^d)$ correct, then the following inequality holds:*

$$\binom{n+\tilde{d}}{\tilde{d}} \geq \sum_{i=1}^{\tilde{d}+1} \binom{p_{q_i}+\tilde{d}}{\tilde{d}} \tag{8}$$

*where $1 \leq \tilde{d} \leq d$ and the right side denotes the sum of any $\tilde{d}+1$ terms. If $m < \tilde{d}+1$, we assume*

$$\sum_{i=1}^{\tilde{d}+1} \binom{p_{q_i}+\tilde{d}}{\tilde{d}} = \sum_{i=1}^{m} \binom{p_1+\tilde{d}}{\tilde{d}}$$

*Proof.* We only give the proof of $m \geq \tilde{d}+1$. The proof of $m < \tilde{d}+1$ is similar and easier. Note that Eq. (2) holds since $(\mathscr{X}, \mathbf{p}, \Pi_n^d)$ is correct. Thus inequality (8) trivially holds for $\tilde{d} = d$.

Consider the case of $\tilde{d} < d$. Suppose $X_{q_1}, X_{q_2}, \ldots, X_{q_{\tilde{d}+1}}$ are $\tilde{d}+1$ arbitrary nodes. Then there exist $d-\tilde{d}$ linearly independent linear polynomial $l_i(X), i = \tilde{d}+1, 2, \ldots, d$ vanishing on these $\tilde{d}+1$ nodes. Assume $l_i(X), i = 1, \ldots, \tilde{d}$ are $\tilde{d}$ linear polynomial such that all $l_i(X), i = 1, 2, \ldots, d$ are linearly independent. Take the following affine transformation

$$T: \quad y_i = l_i(X), i = 1, 2, \ldots, d \tag{9}$$

Let $Y_i = T(X_i), i = 1, 2, \ldots, m$ and $\mathscr{Y} = T(\mathscr{X})$. Thus under the new coordinate system, the last $d-\tilde{d}$ coordinates of $Y_{q_i}, i = 1, 2, \ldots, \tilde{d}+1$ are zero.

Hermite interpolation of type total degree is affinely invariant in the sense that if the interpolation is singular or regular. Hence $(\mathscr{Y}, \mathbf{p}, \Pi_n^d)$ is also correct. Thus for any given $\{c_{q,\boldsymbol{\alpha}}, 0 \leq |\boldsymbol{\alpha}| \leq p_q, 1 \leq q \leq m\}$ there is a unique $f \in \Pi_n^d$ satisfying

$$\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_d}}{\partial y_1^{\alpha_1} \ldots \partial y_d^{\alpha_d}} f(Y_q) = c_{q,\boldsymbol{\alpha}}, \quad 1 \leq q \leq m, \quad 0 \leq |\boldsymbol{\alpha}| \leq p_q \tag{10}$$

Specially, we have

$$\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_{\tilde{d}}}}{\partial y_1^{\alpha_1} \ldots \partial y_{\tilde{d}}^{\alpha_{\tilde{d}}}} f(Y_q) = c_{q,\boldsymbol{\alpha}}, \quad 1 \leq q \leq m, \quad 0 \leq |\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \ldots + \alpha_{\tilde{d}} \leq p_q$$

which means

$$\binom{n+\tilde{d}}{\tilde{d}} \geq \sum_{i=1}^{\tilde{d}+1} \binom{p_{q_i}+\tilde{d}}{\tilde{d}}$$

The proof is completed. $\square$

For convenience, we always order $0 \leq p_1 \leq p_2 \leq \ldots \leq p_m$ in what follows.

**Corollary 1.** *If set $\tilde{d}=1$, then*

$$n+1 \geq p_m + p_{m-1} + 2, \quad \text{or} \quad n \geq p_m + p_{m-1} + 1. \tag{11}$$

The following theorem can be use to judge whether the interpolation scheme is singular for small $m$.

**Theorem 5.** *Given $\mathscr{X} = \{X_1, X_2, \ldots, X_m\}$ and $\mathbf{p} = \{p_1, p_2, \ldots, p_m\}$, if*

$$p_1 + p_2 + \ldots + p_m + m \leq nd, \tag{12}$$

*then Hermite interpolation of type total degree is singular. Here the numbers $p_q$ and $n$ are assumed to satisfy Eq.* (2)*.*

*Proof.* We only need to find a polynomial satisfying the homogenous interpolation condition, which can be done by giving an algorithm for its construction.

> **Step 1.** Set $f(x_1, x_2, \ldots, x_d) = 1$ and $\tilde{\mathbf{p}} = \{\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_m\} := \{p_1+1, p_2+1, \ldots, p_m+1\}$.
>
> **Step 2.** If the number of the nonzero in $\tilde{\mathbf{p}}$ is no more than $d$, let $l(x_1, x_2, \ldots, x_d)$ be a linear polynomial vanishing on $X_1, X_2, \ldots, X_m$. Take $\tilde{p}_{\max} = \max\{\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_m\}$ and set $f = f \cdot l^{\tilde{p}_{\max}}$. Otherwise, go to step 3.
>
> **Step 3.** Suppose $\tilde{p}_{i_1}, \tilde{p}_{i_2}, \ldots, \tilde{p}_{i_d}$ are $d$ largest numbers in $\tilde{\mathbf{p}}$. Clearly, there must exist at least one linear polynomial vanishing at any $d$ points. Denote by $l_{i_1 i_2 \ldots i_d}$ the linear polynomial vanishing on $X_{i_1}, X_{i_2}, \ldots, X_{i_d}$. Set $f = f \cdot l_{i_1 i_2 \ldots i_d}$. Let $\tilde{p}_{i_j} = \tilde{p}_{i_j} - 1, j = 1, 2, \ldots, d$ and $\tilde{p}_i = \tilde{p}_i$ if $i \in \{1, 2, \ldots, m\} / \{i_1, i_2, \ldots, i_d\}$. Go to Step 2.

We want to show that the polynomial $f$ constructed by this algorithm satisfied our requirement. Firstly, to this end we need to show that the algorithm does eventually terminate. Denote $|\tilde{\mathbf{p}}| = \sum_{i=1}^{m} \tilde{p}_i$. The key observation is that $|\tilde{\mathbf{p}}|$ will be dropped by $d$ after step 3. Hence the algorithm will terminate since at the beginning $|\tilde{\mathbf{p}}| \leq nd$.

Next, it is easy to know that this kind of polynomial $f$ constructed by the algorithm satisfies the homogenous interpolation conditions by Theorem 3.

Finally, we need to show that $\deg(f) \leq n$. Assume that Step 3 has been run $t$ times totally. Clearly, $t$ is no more than $n$ according to the assumption of the theorem. To complete the proof, now we estimate the degree of the polynomial $f$. Suppose we are in a situation to run the final step. That is, there are only no more than $d$ nonzero numbers in $\tilde{\mathbf{p}}$. We consider the following three cases.

1) If all the numbers in $\tilde{\mathbf{p}}$ are zeros, then the degree of $f$ is $t$ which is not larger than $n$.

2) The largest number in $\tilde{\mathbf{p}}$ equals to 1, that is, $\tilde{p}_{\max} = 1$. In this case, we have $t \le n-1$, which will leads to $\deg(f) \le n$.

3) If $\tilde{p}_{\max} > 1$, we will show that the degree of $f$ is no more than $k := \max\{p_1+1, p_2+1, \ldots, p_m+1\}$. Clearly, it is enough to show that $\tilde{p}_{max} + t = k$. To this purpose, denote by $\tilde{\mathbf{p}}^{(j)}$ the set $\tilde{\mathbf{p}}$ after running the third Step $j$ times. For convenience, we also write $\{p_1+1, p_2+1, \ldots, p_m+1\}$ as $\tilde{\mathbf{p}}^{(0)}$. Based on these notation we have that $\tilde{p}_{\max}$ is the maximum number in $\tilde{\mathbf{p}}^{(t)}$. Furthermore, there are at most $d$ numbers in $\tilde{\mathbf{p}}^{(t)}$ different from zero. Thus we can deduce that $\tilde{p}_{\max}+1$ must be the maximum number in $\tilde{\mathbf{p}}^{(t-1)}$. If it is not the case, then there must exist $d$ numbers in $\tilde{\mathbf{p}}^{(t-1)}$ are larger than or equals to $\tilde{p}_{\max}$, which will leads to more than $d$ numbers in $\tilde{\mathbf{p}}^{(t)}$ different from zero and furtherly contradict the conclusion above. By a similar discussion, we finally derive that $\tilde{p}_{\max}+t$ is the maximum number in $\tilde{\mathbf{p}}^{(0)}$ which implies that $\tilde{p}_{\max}+t = k$.

By collecting above discussion, we get that $f$ satisfies the interpolation condition and its degree is no more than $n$. Thus the interpolation scheme is singular, which completes the proof. □

**Corollary 2.** *Given $\mathscr{X} = \{X_1, X_2, \ldots, X_m\}$ and $\mathbf{p} = \{p_1, p_2, \ldots, p_m\}$, if Eq.* (12) *holds then there exists a polynomial of degree $\le n$, together with all of its partial derivatives of order up to $p_i$, vanishing at $X_i$ for all $i$.*

Theorem 5 is sharp for small $m$. By corollary 1 and theorem 5, we have

**Theorem 6.** *All Hermite interpolation of type total degree are singular in $\mathbb{R}^d$ with $d \ge 2$ if the number $m$ of nodes satisfies $2 \le m \le d+1$, except for Lagrange case.*

*Proof.* We only need to show that Eq. (12) holds in this case. By Eq. (11) in corollary 1, we have $n \ge p_m + p_{m-1} + 1$, which will lead to

$$\begin{aligned} p_1 + p_2 + \ldots + p_m + m &\le p_1 + p_2 + \ldots + p_m + d + 1 \\ &\le dp_{m-1} + p_m + d + 1 \\ &\le d(p_m + p_{m-1} + 1) \\ &\le nd. \end{aligned}$$

This completes the proof by Theorem 5. □

This result appear in [11] and proved for $d = 2, m = d+1$ in [11] and for $d > 2$ in [11]. The following result is more sharp.

**Corollary 3.** *All Hermite interpolation of type total degree are singular in $\mathbb{R}^d$ with $d \ge k(1 + \frac{1}{p_m})$ if $m \le d+k$.*

*Proof.* We only need to show that Eq. (12) holds in this case. Again by Eq. (11) in Corollary 1, we have $n \geq p_m + p_{m-1} + 1$. Thus,

$$\begin{aligned} p_1 + p_2 + \ldots + p_m + m &\leq p_1 + p_2 + \ldots + p_m + d + k \\ &\leq (d + k - 1)p_{m-1} + p_m + d + k \\ &\leq dp_{m-1} + kp_m + d + k \\ &= dp_{m-1} + k(p_m + 1) + d \\ &= dp_{m-1} + kp_m(1 + \frac{1}{p_m}) + d \\ &\leq dp_{m-1} + dp_m + d \\ &\leq nd. \end{aligned}$$

This completes the proof. □

For true Hermite interpolation, $p_m \geq 1$ which means $1 + \frac{1}{p_m} \leq 2$. Thus we have

**Corollary 4.** *All Hermite interpolation of total degree are singular in $\mathbb{R}^d$ with $d \geq 2k$ if the number $m$ of nodes satisfies $m \leq d + k$.*

From corollary 4, we know that all Hermite interpolation of total degree are singular with $d \geq 4$ if the number of nodes satisfies $m \leq d + 2$. And also from corollary 3, all Hermite interpolation of total degree are singular for $d = 3$ and $m = 5$ if $p_m \geq 2$. We claim that this result is very sharp because the corresponding Hermite interpolation with $d = 3, m = 5$ and $p_1 = p_2 = p_3 = p_4 = p_5 = 1$ is almost regular. The regularity was proved by the method of determinant in [11] and also be shown by example 2 in section 2. Besides, if $p_m \leq 1$ and $p_1 = 0$, Eq. (2) never holds. Therefore, for $d = 3$, only one case can produce regular interpolation scheme.

Now let us consider the case of $d = 2$ and $m = d + 2$. It is well known, interpolating the value of a function and all of its partial derivatives of order up to $p$ at each of the three vertices of a triangle as well as the value of the function and all of its derivatives of order up to $p+1/p-1$ at a fourth point lying anywhere in the interior of the triangle by polynomials from $\Pi^2_{2p+2}/\Pi^2_{2p+1}$ is regular. We will prove that in all other cases, the corresponding Hermite interpolation scheme is singular. This can be done by Lemmas 2 and 3.

**Lemma 2.** *All Hermite interpolations of type total degree on 4 points in $\mathbb{R}^2$ are singular if $0 \leq p_1 \leq p_2 \leq p_3 \leq p_4$, $p_4 > p_2$ and $p_3 > p_1$.*

*Proof.* Suppose the interpolation problem is regular, that is, there is a unique $f(x_1, x_2) \in \Pi^2_n$ satisfying

$$\frac{\partial^{\alpha_1+\alpha_2}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} f(X_q) = c_{q,\boldsymbol{\alpha}}, \quad 1 \leq q \leq 4, \quad 0 \leq |\boldsymbol{\alpha}| \leq p_q. \tag{13}$$

It follows from corollary 1 that $n \geq p_3 + p_4 + 1$.

Denote by $l_{ij} = 0$ the line passing through $X_i$ and $X_j$. Consider the following polynomial

$$f = l_{12}^{p_1+1} l_{34}^{p_3+1} l_{24}^{\max\{p_4-p_3,p_2-p_1\}}.$$

Clearly, $f$ satisfies the homogenous interpolation conditions in Eq. (13) and has the degree of

$$(p_1+1)+(p_3+1)+\max\{p_4-p_3,p_2-p_1\} = \max\{p_1+p_4,p_2+p_3\}+2.$$

To complete the proof, it remains to prove $n$ is not less than the degree of $f$, that is,

$$n \geq \max\{p_1+p_4,p_2+p_3\}+2.$$

According to the discussion above, it is enough to show

$$p_4+p_3+1 \geq \max\{p_1+p_4,p_2+p_3\}+2$$

which is equivalent to

$$(p_4-p_2)+(p_3-p_1) \geq |(p_4-p_2)-(p_3-p_1)|+2. \tag{14}$$

Inequality (14) will be satisfied if $p_4 > p_2$ and $p_3 > p_1$. This completes the proof. □

**Lemma 3.** *All Hermite interpolation of total degree with $d = 2$ and $m = 4$ are singular if the orders satisfy one of the two conditions*

$$\begin{array}{ll} i). & p_1 = p_2 = p_3 \text{ and } p_4 \geq p_1+2, \\ ii). & p_2 = p_3 = p_4 \text{ and } p_4 \geq p_1+2. \end{array}$$

*Proof.* We only give a proof for case i). In this case we have

$$\begin{aligned} p_1+p_2+p_3+p_4+4 &= 3p_3+p_4+4 \\ &\leq 2p_3+2p_4+2 \\ &= 2(p_3+p_4+1) \leq nd \end{aligned}$$

which completes the proof by theorem 5. □

It is well known that uniform Hermite interpolation of type total degree never happen because Eq. (2) in this case does not hold, see [11]. Thus for $m = d+2$, we have

**Theorem 7.** *Consider the problem of Hermite interpolation of type total degree on $m = d+2$ nodes in $\mathbb{R}^d$. Then*

- *For $d = 2$, if $p_1 = p_2 = p_3, p_4 = p_3+1$ or $p_1 = p_2-1, p_2 = p_3 = p_4$, it is almost regular.*
- *For $d = 3$, if $p_1 = p_2 = p_3 = p_4 = p_5 = 1$ and $n = 3$, it is almost regular.*

- *Otherwise, it is singular.*

Let us consider the case of $m = d+3$. From corollary 4, all Hermite interpolation of total degree are singular in one of the following cases: i) $d \geq 6, m = d+3$ and $p_m \geq 1$; ii) $d \geq 5, m = d+3$ and $p_m \geq 2$; iii) $d \geq 4, m = d+3$ and $p_m \geq 3$.

For the case of $d = 5, m = 8$ and $p_m = 1$, it is easy to check Eq. (2) never holds.

Let us turn to the case of $d = 4, m = 7$ and $p_m \leq 2$. Eq. (2) holds only if i) $p_6 = p_7 = 2, p_i = 0, 1 \leq i \leq 5$ and $n = 3$; ii) $p_6 = p_7 = 1, p_i = 0, 1 \leq i \leq 5$ and $n = 2$. For these two interpolation schemes, we need the following result from [13]:

**Theorem 8.** *Multivariate Hermite interpolation of type total degree (8) in $R^d$ with at most $d+1$ nodes having $p_q \geq 1$ is regular a.e. if and only if*

$$p_q + p_r < n$$

*for $1 \leq q, r \leq m, q \neq r$.*

Obviously the interpolation problems considered above are singular by the theorem above.

Consider the case of $d = 3$ and $m = 6$. According to Corollary 1, $n \geq p_6 + p_5 + 1$. Thus if $p_6 - 1 = p_5 = p_4 = p_3 = p_2 = p_1 = p$, then $n \geq 2p + 2$ and

$$\begin{aligned} \sum_{i=1}^{6} \binom{p_i+3}{3} - \binom{n+3}{3} &\leq 5\binom{p+3}{3} + \binom{p+4}{3} - \binom{2p+2+3}{3} \\ &= -\frac{1}{6}(2p+2)(p+2)(p+1) < 0 \end{aligned}$$

which implies that Eq. (2) never holds. Else if $p_6 > p_5 > p_1$, then

$$\begin{aligned} p_1 + p_2 + \ldots + p_6 + 6 &\leq 5p_5 + p_6 + 5 \\ &\leq 3p_5 + 3p_6 + 3 \\ &\leq 3(p_5 + p_6 + 1) \leq 3n. \end{aligned}$$

Thus it follows from Theorem 5 that the Hermite interpolation problem is also singular in this case.

Otherwise, $p_5 = p_6$. In this case

$$\begin{aligned} \sum_{i=1}^{6} \binom{p_i+3}{3} - \binom{n+3}{3} &\leq 6\binom{p_6+3}{3} - \binom{2p_6+4}{3} \\ &= -\frac{1}{3}(p_6-3)(p_6+2)(p_6+1). \end{aligned}$$

Thus Eq. (2) does not hold for $p_5 = p_6 > 3$. Moreover, by a careful check and computation, Eq. (2) also does not hold for $p_5 = p_6 < 3$. As a result, Eq. (2) only hold for $p_i = 3$ and $n = 7$. By Example 2 in section 2, the uniform Hermite interpolation problem is almost regular.

Finally, we consider the case of $d = 2$ and $m = 5$. In this case, we claim that $n < 2p_5 + 2$ if the Hermite interpolation problem is regular. In fact, for any 5 points in the plane, there must exist a non-trival quadratic $Q(x, y)$ vanishing at these points. Let $P(x, y) = [Q(x, y)]^{p_5+1}$. Then $P$, together with all of its partial derivatives of order up to $p_5$, vanish at these points.

**Lemma 4.** *Given* $\mathscr{X} = \{X_1, X_2, \ldots, X_5\} \subset \mathbb{R}^2$ *and* $\mathbf{p} = \{p_1, p_2, \ldots, p_5\}$ *with* $p_1 \leq p_2 \leq p_3 \leq p_4 \leq p_5$*, if Eq.* (2) *holds and*

$$p_2 + p_3 + 2 \leq p_4 + p_5 \tag{15}$$

*then Hermite interpolation of type total degree is singular.*

*Proof.* As discussed above, there exists a non-trival quadratic $Q(x, y)$ vanishing at these five points. Since

$$\begin{aligned} & (p_2 - p_1 - 1) + (p_3 - p_1 - 1) + (p_4 - p_1 - 1) + (p_5 - p_1 - 1) + 4 \\ = \; & p_2 + p_3 + p_4 + p_5 - 4p_1 \leq 2(p_4 + p_5 - 2p_1 - 1), \end{aligned}$$

due to Corollary 2 there exists a polynomial $f(x, y)$ of degree $\leq p_4 + p_5 - 2p_1 - 1$, together with all of its partial derivatives of order up to $p_i - p_1 - 1$(if negative, no interpolation happens), vanishing at $X_i$ for all $2 \leq i \leq 5$. Let $P(x, y) = [Q(x, y)]^{p_1+1} \cdot f(x, y)$. Thus $P$, together with all of its partial derivatives of order up to $p_i$, vanish at $X_i$ for all $i$. It is easy to get that the degree of $P$ is no more than $p_4 + p_5 + 1$. This completes the proof. □

**Lemma 5** ([11])**.** *The uniform Hermite interpolation of type total degree on* 5 *nodes in* $\mathbb{R}^2$ *is singular.*

**Lemma 6.** *If* $p_1 \leq p_2 + 1 = p_3 = p_4 = p_5$ *and Eq.* (2) *holds, then the Hermite interpolation of type total degree is singular.*

*Proof.* Suppose the interpolation problem is regular. Then, $p_4 + p_5 + 1 \leq n < 2p_5 + 2$, that is, $n = 2p_5 + 1$. However,

$$\sum_{i=1}^{5} \binom{p_i + 2}{2} > \sum_{i=2}^{5} \binom{p_i + 2}{2} = \binom{n+2}{2},$$

which contradicts Eq. (2). □

**Lemma 7.** *If* $p_1 \leq p_2 = p_3 = p_4 = p_5 - 1$ *and Eq.* (2) *holds, then the Hermite interpolation of type total degree is singular.*

*Proof.* Suppose the interpolation problem is regular. Then we have

$$p_4 + p_5 + 1 \leq n < 2p_5 + 2$$

and

$$\sum_{i=1}^{5} \binom{p_i + 2}{2} > \sum_{i=2}^{5} \binom{p_i + 2}{2} = \binom{2p_4 + 2 + 2}{2}.$$

Thus, $n = 2p_5 + 1$. Let $Q(x, y)$ be the quadratic polynomial vanishing at these 5 points. Again by the way of Lemma 4, we can get a polynomial of degree no more than $n-(2p_1+2)$, together with all of its partial derivatives of order up to $p_i-p_1-1$, vanishing at $X_i$ for $2 \leq i \leq 5$. Thus $[Q(x, y)]^{p_1+1} \cdot f$ satisfies the homogenous interpolation condition. It is easy to check that

$$2(p_1 + 1) + n - (2p_1 + 2) = n$$

which completes the proof. $\square$

By collecting the discussion above, we have

**Theorem 9.** *All Hermite interpolation of total degree on $m = d+3$ points are singular in $\mathbb{R}^d$ except for the case of $d = 3, n = 7$ and $p_i = 3$ for $i = 1, 2, \ldots, 6$.*

Combing Corollary 4 and Theorems 6,7,9, we have proved

**Theorem 10.** *Consider the problem of Hermite interpolation of type total degree on $m = d + k$ nodes in $\mathbb{R}^d$. Then*

1. *For $k \leq 1$, it is singular.*
2. *For $k = 2$, if $d = 2$, and $p_1 = p_2 = p_3, p_4 = p_3 + 1$ or $p_1 = p_2 - 1, p_2 = p_3 = p_4$, it is almost regular; else if $d = 3$, and $p_i = 1, i = 1, 2, 3, 4, 5, n = 3$, it is almost regular; otherwise it is singular.*
3. *For $k = 3$, if $d = 3$ and $p_i = 3, (i = 1, 2, \ldots, 6), n = 7$, it is almost regular; otherwise it is singular.*
4. *If $d \geq 2k$, it is singular.*

# 4 Conclusion

In this paper, we consider the singular problem of multivariate Hermite interpolation of total degree. We make a detailed investigation for Hermite interpolation problem of type total degree on $m = d + k$ nodes in $\mathbb{R}^d$. Our results imply that the interpolation problem in $\mathbb{R}^d$ is singular on $m = d + k$ nodes for $d \geq 2k$. For $k \leq 3$, it is shown that only very few cases can produce regular interpolation. The method developed in this paper can deal with the case of small $k$ compared to $d$. For bigger $k$, Theorem 5 is not very sharp.

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